JOURNAL OF APPROXIMATION THEORY 38, 344–353 (1983)

Best Conditions for the Norm Convergence of Fourier Series*

DONALD I. CARTWRIGHT† AND PAOLO M. SOARDI

Istituto Matematico, Università di Milano, Via C. Saldini 50, 20133 Milan, Italy

Communicated by F. L. Bauer

Received March 15, 1982

1. INTRODUCTION

Suppose that E is either the space $C(\mathbb{T}^N)$ or $L^1(\mathbb{T}^N)$, $N \geqslant 1$. It is known that E does not admit convergence in norm for Fourier series over squares or spheres. On the other hand, it is known that if the kth modulus of continuity of a function $f \in E$ satisfies the condition

$$\omega_k(t, f) = o(|\log(t)|^{-N}) \qquad \text{as} \quad t \to 0 \tag{1}$$

in the case of squares (for some $k \geqslant 1$), or if $N \geqslant 2$,

$$\omega_k(t, f) = o(t^{(N-1)/2}) \qquad \text{as} \quad t \to 0 \tag{2}$$

in the case of spheres (for some $k \geqslant (N-1)/2$), then the Fourier series of f converges to f in the norm of E. As a special case of a theorem of Zhizhiashvili [15, 16] one knows that (1) is the best condition on f assuring norm convergence of its Fourier series over squares, in the sense that the small o cannot be replaced by large O.

In this paper we first show that in the same sense (2) is the best possible condition on f assuring norm convergence of its Fourier series over spheres. In the case of uniform convergence, Il'in (see [1]) had, amongst other things, proved the following weaker result: if f is in the Hölder class $C^\alpha(\mathbb{T}^N)$ for $\alpha = (N-1)/2$ (this implies (2)), then the Fourier series of f over spheres converges uniformly to f. Furthermore, the index $(N-1)/2$ is sharp. (See [1] for the definition of $C^\alpha(\mathbb{T}^N)$.)

* Research partially supported by the C.N.R.

† Permanent address: Department of Pure Mathematics, University of Sydney, Sydney 2006, N.S.W., Australia.

0021-9045/83 $3.00

Our method is quite general, and may be used to obtain analogous results for the divergence of Fourier series over squares or cylinders, for example. Indeed, we are also able to obtain analogous results in the context of compact Riemannian symmetric spaces of rank one. One of the tools of our proof is an analogue of the classical Bernstein inequality. It will be evident from the proof that the restriction to the rank one case is necessary only because of the current lack of good estimates for the relevant Lebesgue constants.

2. Multiple Fourier Series

In this section we generally adopt the notation of [12], and use [9] as a general reference. We consider spherical partial sums

$$S_r(f)(x) = \sum_{|m| \leqslant r} \hat{f}(m)\, e^{2\pi i m \cdot x} = (D_r * f)(x),$$

where

$$D_r(x) = \sum_{|m| \leqslant r} e^{2\pi i m \cdot x},$$

these sums being extended over $\{m = (m_1, \ldots, m_N) \in \mathbb{Z}^N\colon\ |m| = (m_1^2 + \cdots + m_N^2)^{1/2} \leqslant r\}$. If $E = C(\mathbb{T}^N)$ or $L^1(\mathbb{T}^N)$ and if $s \in \{0, 1, \ldots\}$, we say that $f \in E^{(s)}$ if for each multi-index $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{Z}_+^N$ with $\beta_1 + \cdots + \beta_N = s$, the partial derivative $D^\beta f$ exists in the norm sense. We then write $\omega_k(t, f^{(s)})$ for the sum of the moduli of continuity $\omega_k(t, D^\beta f)$ over all such β.

The letter C denotes a positive constant which may vary from line to line.

THEOREM A. *Assume that* $N \geqslant 2$.

(a) *Suppose that* $f \in E$ *and that* (*for some* $k > (N-1)/2$)

$$\omega_k(t, f) = o(t^{(N-1)/2}) \qquad as \quad t \to 0.$$

Then

$$D_r * f \to f \qquad in\ E\ as\ r \to \infty.$$

(b) *There is a function* $F \in E$ *such that* (*for all* $k > (N-1)/2$) $\omega_k(t, F) = O(t^{(N-1)/2})$ *as* $t \to 0$, *but* $D_r * F$ *does not converge to* F. *In fact,* F *can be chosen so that* $F \in E^{(s)}$ *and* $\omega_2(t, F^{(s)}) = O(t^{(N-1)/2 - s})$, *where* s *is the largest integer* $< (N-1)/2$.

Proof. Babenko (see [1] for a proof) has shown that there are constants $C_1, C_2 > 0$ such that

$$C_1 r^{(N-1)/2} \leqslant \|D_r\|_1 \leqslant C_2 r^{(N-1)/2} \qquad \text{for all} \quad r > 0. \tag{3}$$

The proof of (a) may be obtained in the standard way from the multidimensional Jackson approximation theorem and the right-hand inequality in (3).

For the proof of (b), first pick a number $a > 1$ such that $C_1 a^{(N-1)/2} > C_2$. Since $\sup\{\|g * f\| : f \in E, \|f\| \leqslant 1\} = \|g\|_1$ for $g \in L^1(\mathbb{T}^N)$, we can, for each $r > 0$, pick $f_r \in E$ with $\|f_r\| \leqslant 1$ and $\|(D_{ar} - D_r) * f_r\| \geqslant Cr^{(N-1)/2}$. Now take any ϕ in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ such that $\hat{\phi}(\xi) = 1$ if $1 \leqslant |\xi| \leqslant a$ and $\hat{\phi}(\xi) = 0$ if $|\xi| \notin (\frac{1}{2}, 2a)$. For each $r > 0$ define s_r on $\mathbb{T}^N$ by

$$s_r(t) = \sum_{m \in \mathbb{Z}^N} \hat{\phi}\left(\frac{m}{r}\right) e^{2\pi i m \cdot t}.$$

Then $\|s_r\|_1 \leqslant \|\phi\|_1$ follows from the Poisson summation formula. For each $r > 0$ we set $g_r = f_r * s_r$. It is clear that

$$\|g_r\| \leqslant C, \tag{4}$$

$$\|(D_{ar} - D_r) * g_r\| \geqslant Cr^{(N-1)/2} \tag{5}$$

and

$$\hat{g}_r(m) = 0 \qquad \text{unless} \quad \frac{r}{2} < |m| < 2ar. \tag{6}$$

Now write $q = 4a$ and G_j for g_{q^j}. Define $F \in E$ by

$$F = \sum_{j=0}^{\infty} q^{-j(N-1)/2} G_j.$$

It follows from (4) that F is well defined, while from (5) and (6) that

$$\|(D_{aq^j} - D_{q^j}) * F\| = \|(D_{aq^j} - D_{q^j}) * G_j\| \, q^{-j(N-1)/2} \geqslant C.$$

Thus $D_r * F$ does not converge to F in E.

Now for any multi-index $\beta = (\beta_1, ..., \beta_N) \in \mathbb{Z}_+^N$ with $\beta_1 + \cdots + \beta_N = s$, where s is defined above, Bernstein's inequality and (6) show that $\|D^\beta G_j\| \leqslant Cq^{js}$. It follows that $F \in E^{(s)}$. Now let $0 < t < 1$ and let M be the largest integer such that $q^M \leqslant t^{-1}$. Using the estimate $\|\Delta_h^2(D^\beta G_j)\| \leqslant C|h|^2 q^{j(2+s)}$ for $j \leqslant M$, which follows from Bernstein's inequality, and the estimate $\|\Delta_h^2(D^\beta G_j)\| \leqslant Cq^{js}$ for $j > M$, and the fact that $2 + s > (N-1)/2$, we see that

$$\begin{aligned}\|\Delta_h^2(D^\beta F)\| &\leqslant C\,|h|^2 \sum_{j=0}^{M} q^{j(2+s-(N-1)/2)} + C \sum_{j=M+1}^{\infty} q^{-j((N-1)/2-s)} \\ &\leqslant C\,|h|^2\, q^{M(2+s-(N-1)/2)} + Cq^{-M((N-1)/2-s)} \\ &\leqslant Ct^{(N-1)/2-s} \qquad \text{if} \quad |h| \leqslant t.\end{aligned}$$

Thus $\omega_2(t, F^{(s)}) \leqslant Ct^{(N-1)/2-s}$.

Remarks. (i) If N is even, then $s = [(N-1)/2]$, and we may replace $\omega_2(t, F^{(s)})$ by $\omega_1(t, F^{(s)})$, still obtaining $\omega_1(t, F^{(s)}) = O(t^{1/2})$. For N odd, so that $s = (N-1)/2 - 1$, the technique of the proof yields the estimate $\omega_1(t, F^{(s)}) = O(t\,|\log t|)$, so that we must use ω_2, which satisfies $\omega_2(t, F^{(s)}) = O(t)$.

(ii) Let us indicate the modifications to the above proof which we can make to obtain the analogous result for the divergence of Fourier series over squares. The estimate $\|D_{ar} - D_r\|_1 \geqslant C(\log r)^N$ clearly holds for $a = 2$, D_r now denoting the Dirichlet kernel $\sum e^{2\pi i m \cdot x}$, the sum being over $\{m = (m_1, ..., m_N) \in \mathbb{Z}^N\colon |m_j| \leqslant r$ for each $j\}$. The function F is defined by $F = \sum_{j=0}^{\infty} q^{-jN} G_j$, where $G_j = g_r$ for $r = q^{q^j}$, and $q = 4a$ as before. We obtain the estimate $\omega_1(t, F) = O(|\log t|^{-N})$ as in the above proof.

With slight modifications the proof may also be applied to Fourier series over "cylinders," for which D_r is the sum $\sum e^{2\pi i m \cdot x}$ over $\{m = (m_1, ..., m_N) \in \mathbb{Z}^N\colon (m_1^2 + \cdots + m_K^2)^{1/2} \leqslant r$ and $|m_j| \leqslant r$ for $j = K+1, ..., N\}$, where $2 \leqslant K < N$. Then $\|D_r\|_1 \sim r^{(K-1)/2}(\log r)^{N-K}$, and we define $F = \sum_{j=0}^{\infty} G_j q^{-j(K-1)/2} j^{-(N-K)}$, with G_j as in the proof of the theorem.

3. Compact Symmetric Spaces of Rank One

A general reference for this section is [4].

Let M be a compact Riemannian symmetric space of rank one. We may write $M = G/K$, where G is a compact connected simply connected semisimple Lie group and K is a closed subgroup of G. In the usual way, we fix a maximal torus T in G and construct the semi-lattice Λ^+ of dominant weights relative to a choice of $\mathfrak{R}^+$, the set of positive roots relative to T. Under the correspondence between Λ^+ and the dual object $\hat{G}$ of G, it is known (see, e.g., [14]) that the representations of G of class 1 correspond to the multiples $n\rho$, $n = 0, 1, ...$, of a fixed $\rho \in \Lambda^+$.

Let $f \in E$, where $E = C(M)$ or $L^1(M) = L^1(M, \mu)$, where μ is the normalized G-invariant measure on M. We are interested in stating best conditions for the norm convergence of the expansions

$$f \sim \sum_{n=0}^{\infty} f * Z^n,$$

where $*$ is the convolution on $L^1(M)$ induced from that on G, and where Z^n is the zonal harmonic function corresponding to np. These conditions will be given in terms of the moduli of continuity $\omega_k(t, f)$ of f, defined by $\omega_k(t, f) = \sup\{\|\Delta_h^k f\|: d(e, h) \leqslant t\}$. Here d denotes the geodesic distance on G, $\| \ \|$ the norm on E and, for $h \in G$ and $\xi \in M$,

$$(\Delta_h^k f)(\xi) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(h^{-j} \cdot \xi).$$

We shall also need the quantities $\omega_k(t, f^{(s)})$, defined as follows (see [6]): For $X \in \mathfrak{g}$, the Lie algebra of G, and $f \in E$, we define $Xf: M \to \mathbb{C}$ by

$$(Xf)(\xi) = \lim_{t \to 0} \frac{f(\exp(-tX) \cdot \xi) - f(\xi)}{t},$$

provided that the limit exists in the norm of E. We let $E^{(s)}$ denote the set of $f \in E$ for which $Y_1 Y_2 \cdots Y_s f$ exists for any $Y_1, \ldots, Y_s \in \mathfrak{g}$, and for $f \in E^{(s)}$ we define

$$\omega_k(t, f^{(s)}) = \sum_{i_1, \ldots, i_s = 1}^{n} \omega_k(t, X_{i_1} X_{i_2} \cdots X_{i_s} f),$$

where $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$.

We need two lemmas, which we state in the context of a compact connected simply connected semisimple Lie group G. We denote by d_λ and χ_λ the dimension and character, respectively, of the representation corresponding to $\lambda \in \Lambda^+$. We write β for the weight $\frac{1}{2} \sum_{\alpha \in \mathfrak{R}^+} \alpha$ and $|\lambda|$ for the norm of $\lambda \in \Lambda^+$ induced by the Killing form on $\mathfrak{g}$.

LEMMA 1. *Let $a > 1$ and let Φ be an infinitely differentiable function on $(0, \infty)$ such that $\Phi(x) = 1$ if $x \in [1, a]$ and $\Phi(x) = 0$ if $x \notin (\frac{1}{2}, 2a)$. For each $R > 0$ let*

$$s_R = \sum_{\lambda \in \Lambda^+} \Phi\left(\frac{|\lambda + \beta|}{R}\right) d_\lambda \chi_\lambda.$$

Then there is a constant C such that $\|s_R\|_1 \leqslant C$ for all $R > 0$.

Proof. It is evident that our function Φ satisfies the conditions of Théorème 1 of Clerc [3]. We may therefore write

$$s_R(\exp H) = \frac{CR^n}{D(\exp H)} \sum_{\zeta \in \mathfrak{t}_e} \left(\prod_{\alpha \in \mathfrak{R}^+} \alpha(H+\zeta) \right) \psi(R\,|H+\zeta|), \tag{7}$$

in the notation of [3], and in our case the function ψ must satisfy a condition $|\psi(r)| \leqslant Cr^{-n-\varepsilon}$. We now use the Weyl formula $\|s_R\|_1 = C\int_Q |s_R(\exp H)|\,|D(\exp H)|^2\,dH$, where $Q \subseteq \mathfrak{t}$ is a fundamental domain centred at 0, and modify the proof of Théorème 3 in [3]. The integral over $\{H \in Q\colon |H| < 1/R\}$ is bounded because of the inequality $|s_R(x)| \leqslant CR^n$, which follows easily from the definition of s_R. When $|H| \geqslant 1/R$ we consider first the integral of the term in (7) corresponding to $\zeta = 0$. To bound this we use the obvious inequality $|CR^n\,(\prod_{\alpha \in \mathfrak{R}^+} \alpha(H))\,\psi(R\,|H|)\,D(\exp H)| \leqslant CR^n\,|H|^m\,(R\,|H|)^{-n-\varepsilon}\,|H|^m = CR^{-\varepsilon}\,|H|^{-l-\varepsilon}$. To deal with the remaining terms of (7), we notice that there are constants $C_1, C_2 > 0$ such that $C_1\,|\zeta| \leqslant |H+\zeta| \leqslant C_2\,|\zeta|$ for all $H \in Q$ and all non-zero $\zeta \in \mathfrak{t}_e$. Using $|\psi(r)| \leqslant Cr^{-n}$, it is immediate that the relevant integral is bounded by $C\sum_{\zeta \neq 0} (1/|\zeta|^{l+m})$, independent of R.

In the next lemma we denote by $\mathrm{Trig}_R(G)$ the linear span of the trigonometric polynomials on G corresponding to the weights $\lambda \in \Lambda^+$ with $|\lambda + \beta| \leqslant R$. For functions f on G and $X \in \mathfrak{g}$, the function Xf is defined in the same way as for functions on M. The authors are indebted here to Michael Cowling for the key idea of the proof.

LEMMA 2 ("Bernstein's inequality"). *If $f \in \mathrm{Trig}_R(G)$, $X \in \mathfrak{g}$ and $1 \leqslant p \leqslant \infty$, then Xf exists, is in $\mathrm{Trig}_R(G)$, and satisfies*

$$\|Xf\|_p \leqslant C_G\,\|X\|\,R\,\|f\|_p$$

for some constant $C_G > 0$.

Proof. It is easy to see that Xf exists and is in $\mathrm{Trig}_R(G)$. To derive the inequality, we first suppose that $X \in \mathfrak{t}$, the Lie algebra of T. Let $X_1, \ldots, X_l \in \mathfrak{t}$ correspond under an isomorphism $T \cong \mathbb{T}^l$ to the usual partial derivatives. If now $f \in \mathrm{Trig}_R(G)$, then $g = f|_T$ is a linear combination of the characters ξ_μ of T corresponding to the weights μ in the saturated hulls of the dominant weights λ with $|\lambda + \beta| \leqslant R$. But ξ_μ corresponds to an exponential $e^{i(m_1 t_1 + \cdots + m_l t_l)}$ for integers m_j, $1 \leqslant j \leqslant l$, satisfying

$$\begin{aligned} |m_j| = |\mu(X_j)| &\leqslant |\mu|\,|X_j| \\ &\leqslant C\,|\lambda + \beta| \qquad \text{(see [5, Lemma 13.4 C])} \\ &\leqslant CR. \end{aligned}$$

Writing $X = a_1 X_1 + \cdots + a_l X_l$, it follows from the classical Bernstein inequality that

$$\|Xg\|_p \leqslant \sum_{j=1}^{l} |a_j| \, \|X_j g\|_p \leqslant CR \left(\sum_{j=1}^{l} |a_j| \right) \|g\|_p \leqslant CR \, |X| \, \|g\|_p,$$

where $\| \ \|_p$ denotes the norm in $L^p(T)$ in the preceding line only.

If μ is the normalized G-invariant measure on the left coset space G/T and if $x \to \dot{x}$ denotes the canonical map $G \to G/T$, then applying the above calculations to $g_x : y \to f(xy)$, $y \in T$, we have

$$\begin{aligned}
\|Xf\|_p^p &= \int_G |(Xf)(x)|^p \, dx \\
&= \int_{G/T} \int_T |(Xg_x)(y)|^p \, dy \, d\mu(\dot{x}) \\
&\leqslant (CR\,|X|)^p \int_{G/T} \int_T |g_x(y)|^p \, dy \, d\mu(\dot{x}) \\
&= (CR\,|X| \, \|f\|_p)^p.
\end{aligned}$$

In the case of a general $X \in \mathfrak{g}$, there is an $x_0 \in G$ and a $Y \in \mathfrak{t}$ such that $X = \mathrm{Ad}(x_0) Y$ [4, Theorem V.6.4]. Then $(Xf)(x) = (Yh)(x_0^{-1} x)$ holds for $h(x) = f(x_0 x)$. Thus $\|Xf\|_p = \|Yh\|_p \leqslant CR\,|Y| \, \|h\|_p = CR\,|X| \, \|f\|_p$.

Remark. In the case $p = \infty$, the last result was obtained by Ragozin [10] using other methods.

In the statement of our next theorem, we shall have in mind the well-known correspondence between the zonal harmonics Z^n and normalized Jacobi polynomials $R_n^{\alpha,\beta} = P_n^{\alpha,\beta}/P_n^{\alpha,\beta}(1)$ for suitable $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ (see, e.g., [2]). We shall exclude the case $\alpha = -\frac{1}{2}$, as it corresponds to the case $M = \mathbb{T}^1$. See [13] for general facts about Jacobi polynomials.

THEOREM B. *Let M be a compact symmetric space of rank one and dimension $d \geqslant 2$, and let $\alpha = (d-2)/2$.*

(a) *If $f \in E^{(s)}$ and $\omega_2(t, f^{(s)}) = o(t^{\alpha - s + 1/2})$ as $t \to 0$, for some $s \geqslant 0$, then $\sum_{n=0}^{N} f * Z^n \to f$ in E as $N \to \infty$.*

(b) *On the other hand, there exists a function $F \in E^{(s)}$, where s is the largest integer $< \alpha + \frac{1}{2}$, which may be chosen zonal, such that $\omega_2(t, F^{(s)}) = O(t^{\alpha - s + 1/2})$ as $t \to 0$, but such that $\sum_{n=0}^{N} F * Z^n$ does not converge to F in E.*

Proof. For zonal $g \in L^1(M)$ we have

$$\begin{aligned}
&\sup\{\|f * g\| : f \in E \text{ and } \|f\| \leqslant 1\} \\
&\qquad = \|g\|_1 = \sup\{\|f * g\| : f \in E, f \text{ zonal, and } \|f\| \leqslant 1\}. \qquad (8)
\end{aligned}$$

Taking $g = D_r = \sum_{n=0}^{r} Z^n$, we have

$$\|D_r\|_1 = \int_{-1}^{1} \left| \sum_{n=0}^{r} h_n R_n^{\alpha,\beta}(x) \right| (1-x)^\alpha (1+x)^\beta \, dx,$$

where

$$h_n^{-1} = \int_{-1}^{1} (R_n^{\alpha,\beta}(x))^2 (1-x)^\beta \, dx.$$

We know [7, 11] that for suitable $C_1, C_2 > 0$ we have

$$C_1 r^{\alpha+1/2} \leqslant \|D_r\|_1 \leqslant C_2 r^{\alpha+1/2}. \tag{9}$$

Now using Johnen's Jackson-type theorem [6, Folgerung 4.5] and the right-hand side of (9), it is clear that (a) holds.

To prove (b), we pick an integer a such that $a^{\alpha+1/2} C_1 > C_2$. In view of (8) and (9) we can choose a zonal function $f_r \in E$ for each $r \in \mathbb{N}$ such that $\|f_r\| \leqslant 1$ and $\|f_r * (D_{ar} - D_r)\| \geqslant C r^{\alpha+1/2}$. By Lemma 1 there is for each $r \in \mathbb{N}$ a central function p_r on G such that its Fourier transform $\hat{p}_r$ satisfies $\hat{p}_r(k\rho) = 1$ for $r < k \leqslant ar$ and $\hat{p}_r(k\rho) = 0$ if $k \geqslant 2ar$ or $k < r/2$, and such that $\|p_r\|_1 \leqslant C$. Let g_r be the function on M corresponding to $p_r * \tilde{f}_r$, where $\tilde{f}_r$ is the function on G corresponding to f_r. Then, writing $\hat{g}_r(k)$ for $(p_r * \tilde{f}_r)\hat{}\ (k\rho)$, we have

$$\|g_r\| \leqslant C, \tag{10}$$

$$\|g_r * (D_{ar} - D_r)\| \geqslant C r^{\alpha+1/2}, \tag{11}$$

and

$$\hat{g}_r(k) = 0 \qquad \text{if} \quad k \notin \left(\frac{r}{2}, 2ar\right]. \tag{12}$$

As in the proof of Theorem A we write $q = 4a$, $G_j = g_{q^j}$ and define $F \in E$ by

$$F = \sum_{j=0}^{\infty} q^{-j(\alpha+1/2)} G_j.$$

By (10), F is well-defined, while (11) and (12) show that $\|F * (D_{aq^j} - D_{q^j})\| = \|G_j * (D_{aq^j} - D_{q^j})\| \, q^{-j(\alpha+1/2)} \geqslant C$, so that $F * D_r$ does not converge to F. Furthermore, F is zonal, since each f_r is zonal and each p_r is central. By Lemma 2, $\|X_{i_1} X_{i_2} \cdots X_{i_s} G_j\| \leqslant C q^{js}$ holds for any $i_1, \ldots, i_s \in \{1, \ldots, n\}$, where $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}$. Thus $F \in E^{(s)}$, as $s < \alpha + \frac{1}{2}$. Using Lemma 2 again and the straightforward inequality $\|\Delta_h f\| \leqslant \|Xf\|$ for $h = \exp X$, we see that $\|\Delta_h^2 (X_{i_1} X_{i_2} \cdots X_{i_s} G_j)\| \leqslant C d(e, h)^2 q^{j(s+2)}$. It follows as in the proof of Theorem A that $\omega_2(t, F^{(s)}) = O(t^{\alpha - s + 1/2})$.

Remark. (i) If d is even, then we may replace ω_2 by ω_1, cf. the first remark after Theorem A.

(ii) Taking as an example $M = SU(2) = SO(4)/SO(3)$, our theorem shows that $\omega_2(t, f) = o(t)$ is the best possible condition for norm convergence of the Fourier series of a function $f \in C(SU(2))$ or $L^1(SU(2))$. See [8] for related results.

(iii) For M as in Theorem B we are able to provide sharp estimates for the convolution norm of D_r acting on the zonal L^p functions. We can thus obtain a theorem about the L^p convergence of the zonal harmonic expansions of such functions. This work will appear elsewhere.

REFERENCES

1. SH. A. ALIMOV, V. A. IL'IN, AND E. M. NIKISHIN, Convergence problems of multiple trigonometric series and spectral decompositions, I and II, *Uspehi Mat. Nauk* **31** (1976), 28–83; **32** (1977), 107–130. [English translation: *Russian Math. Surveys* **31** (1976), 29–86; **32** (1977), 115–139.]
2. R. ASKEY AND N. H. BINGHAM, Gaussian processes on compact symmetric spaces, *Z. Wahrsch. Verw. Gebiete* **37** (1976), 127–143.
3. J.-L. CLERC, Sommes de Riesz et multiplicateurs sur un groupe de Lie compact, *Ann. Inst. Fourier (Grenoble)* **24** (1974), 149–172.
4. S. HELGASON, "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, 1978.
5. J. E. HUMPHREYS, "Introduction to Lie Algebras and Representation Theory," Springer–Verlag, Berlin, 1972.
6. H. JOHNEN, Sätze vom Jackson-Typ auf Darstellungsräumen kompakter zusammenhängender Liegruppen, *in* "Linear Operators and Approximation (Proceedings, Conference at Oberwolfach, 1971," pp. 254–272, Internat. Ser. Numer. Math. Vol. 20, Birkhäuser, Basel, 1972.
7. L. LORCH, Lebesgue constants for Jacobi series, *Proc. Amer. Math. Soc.* **10** (1959), 756–761.
8. R. A. MAYER, Fourier series of differentiable functions on $SU(2)$, *Duke Math. J.* **34** (1967), 549–554.
9. S. M. NIKOL'SKII, "Approximation of Functions of Several Variables and Imbedding Theorems," Springer–Verlag, Berlin, 1975.
10. D. L. RAGOZIN, Polynomial approximation on compact manifolds and homogeneous spaces, *Trans. Amer. Math. Soc.* **150** (1970), 41–53.
11. H. RAU, Über die Lebesgueschen Konstanten der Reihenentwicklungen nach Jacobischen Polynomen, *J. Reine Angew. Math.* **161** (1929), 237–254.
12. E. M. STEIN AND G. WEISS, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, N.J., 1971.
13. G. SZEGÖ, "Orthogonal Polynomials," Amer. Math. Soc. Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, R.I., 1967.
14. L. VRETARE, Elementary spherical functions on symmetric spaces, *Math. Scand.* **39** (1976), 343–358.

15. L. V. ZHIZHIASHVILI, On the divergence of Fourier series, *Dokl. Akad. Nauk SSSR* **194** (1970), 758–759. [English translation: *Soviet Math. Dokl.* **11** (1970), 1281–1283.]
16. L. V. ZHIZHIASHVILI, Convergence and divergence of Fourier series, *Dokl. Akad. Nauk SSSR* **199** (1971), 1234–1236. [English translation: *Soviet Math. Dokl.* **12** (1971), 1261–1263.]